# Perturbation theory for nonlinear equations

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**Abstract**

For a wide class of nonlinear equations a perturbative solution is constructed. This class includes equations of motion of field theories. The solution possesses a graphical representation in terms of diagrams. To illustrate the formalism we consider the Yang-Mills field equations.

## 1 Introduction

Let $V$ and $W$ be vector spaces, and let

$$F(u) = 0 \tag{1}$$

be an equation, where $F : V \to W$ is a given function, and $u \in V$ is an unknown vector. For a given $u_0 \in V$ one can expand the equation in powers of $v = u - u_0$,

$$F_0 + \sum_{n=1}^{\infty} F_n(v) = 0. \tag{2}$$

Here $F_0 = F(u_0)$, and for $n \geq 1$

$$F_n(v) = \frac{1}{n!} \frac{d^n F(u_0 + \xi v)}{d^n \xi}\bigg|_{\xi=0}. \tag{3}$$

Let $v_0$ be the general solution of the free equation

$$F_1(v) = 0, \tag{4}$$

and let $\mathcal{E}(f)$ be the specific solution of the inhomogeneous equation

$$F_1(v) = -f, \quad f \in W, \tag{5}$$

which is linear in $f$, $\mathcal{E}(af) = a\mathcal{E}(f), a \in \mathbb{R}$. Equation (2) then becomes

$$v = w_0 + \sum_{n=2}^{\infty} p_n(v), \tag{6}$$

where $w_0 = v_0 + p_0$, $p_n = \mathcal{E}(F_n)$.

Equation (1) is rather general. It appears in many problems. In particular, equations of motion of field theories are written in the form (1). For most of nonlinear equations in mathematical physics solutions of equations (4) and (5) are well known [1].

The aim of the present paper is to construct a solution of equation (6).The solution possesses a graphical representation in terms of diagrams.To illustrate the formalism we consider the pure Yang-Mills field equations in the Lorentz gauge.A solution of the equations of motion in spinor electrodynamics was found in ref.[2].

The paper is organized as follows. In the next section we construct a solution of equation (6) and describe its graphical representation. In Sec.3 we consider the Yang-Mills field equations.

## 2 A solution of nonlinear equations

We shall need functions

$$\langle \ldots \rangle_n : V^n \to V, \quad n = 2, 3, \ldots,$$

defined for $v_1, \ldots, v_n \in V$ by

$$\langle v_1, \ldots, v_n \rangle_n = \sum_{r=1}^{n} (-1)^{n-r} \sum_{i_1 < \ldots < i_r} p_n(v_{i_1} + \ldots + v_{i_r}). \tag{7}$$

One can show [3] that $\langle v_1, \ldots, v_n \rangle_n$ is an $n-$ linear symmetric function.

From (7) it follows

$$\langle v, \ldots, v \rangle_n = n! p_n(v),$$

Then equation (6) takes the form

$$v = w_0 + \sum_{n=2}^{\infty} \frac{1}{n!} \langle v, \ldots, v \rangle_n. \tag{8}$$

For $I = (i_1, \ldots, i_n)$ we denote $v_I = (v_{i_1}, \ldots, v_{i_n})$. To solve equation (8) we introduce a family of functions

$$\langle \ldots \rangle : V^m \to V, \qquad m = 2, 3, \ldots,$$

recursively defined by

$$\langle v_1, \ldots, v_m \rangle = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I_1 \cup \ldots \cup I_n = (1 \ldots m)} \langle \langle v_{I_1} \rangle, \langle v_{I_2} \rangle, \ldots, \langle v_{I_n} \rangle \rangle_n, \tag{9}$$

where $I_i, i = 1, \ldots, n$, is an increasing multi-index [1], $I_i \cap I_j = \emptyset$, $\langle v \rangle = v$, and for $I = (i_1, \ldots, i_n)$ $\langle v_I \rangle = \langle v_{i_1}, \ldots, v_{i_n} \rangle$. It is easy to prove by induction that $\langle v_1, \ldots, v_m \rangle$ is an $m-$ linear symmetric function.

For $m = 2$ and $m = 3$ we have

$$\langle v_1, v_2 \rangle = \langle v_1, v_2 \rangle_2,$$

$$\langle v_1, v_2, v_3 \rangle = \langle \langle v_1, v_2 \rangle_2, v_3 \rangle_2 + \langle \langle v_1, v_3 \rangle_2, v_2 \rangle_2 + \langle \langle v_2, v_3 \rangle_2, v_1 \rangle_2 + \langle v_1, v_2, v_3 \rangle_3.$$

Let $P^m_{i_1 \ldots i_n} : V^m \to V^{m-n+1}, m \geq 2, 1 \leq i_1 < \ldots < i_n \leq m$, be defined by

$$P^m_{i_1 \ldots i_n}(v_1, \ldots, v_m) = (\langle v_{i_1}, \ldots v_{i_n} \rangle_n, v_1, \ldots, \widehat{v}_{i_1}, \ldots, \widehat{v}_{i_n}, \ldots, v_m),$$

where $\widehat{v}$ means that $v$ is omitted. If $v \in V$ is given by

$$v = P^{n_s}_{I_s} \ldots P^{m-n_1+1}_{I_2} P^m_{I_1}(v_1, \ldots, v_m) \tag{10}$$

for some $I_1 = (i^1_1, \ldots, i^1_{n_1}), I_2 = (i^2_1, \ldots, i^2_{n_2}), \ldots, I_s = (i^s_1, \ldots, i^s_{n_s}), n_1 + \ldots + n_s - s + 1 = m$, we say that $v$ is a descendant of $(v_1, \ldots, v_m)$.

The functions $\langle v_1, v_2 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are given by the sums of all the descendants of their arguments. Assume that $\langle v_1, \ldots, v_k \rangle, k < m$, is given by

[1] The multi-index $I = (i_1, \ldots, i_n)$ is said to be increasing if $i_1 < \ldots < i_n$.

the sum of all the descendants of $(v_1, \ldots, v_k)$. Each descendant of $(v_1, \ldots, v_m)$ can be written as

$$\langle d_1(v_{I_{i_1}}), d_2(v_{I_{i_2}}), \ldots, d_n(v_{I_{i_n}})\rangle_n, \tag{11}$$

for some $n \geq 2$ and $I_{i_1} \cup \ldots \cup I_{i_n} = (1, \ldots, m)$, where $I_{i_k}$ is an increasing multi-index, $d_k(v_{I_{i_k}})$ is a descendant of $v_{I_{i_k}}, k = 1, \ldots, n$. It is easy to show that for $k \neq l$ $I_{i_k} \cap I_{i_l} = \emptyset$. Then summing all the different functions (11) we get the right-hand side of (9). Thus, we have proved that $\langle v_1, \ldots, v_m\rangle$ is given by the sum of all the descendants of $(v_1, \ldots, v_m)$.

Each descendant can be represented by a diagram. In this diagram an element of $V$ is represented by the line segment ——. A product $\langle v_1, \ldots, v_n\rangle_n$ is represented by the vertex joining the line segments for $v_1, \ldots, v_n$ and $\langle v_1, \ldots, v_n\rangle_n$. The general rule should be clear from Figure 1.Here we show the diagram for

$$P^m_{ij}(v_1, \ldots, v_m) = (\langle v_i, v_j\rangle_2, v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_m).$$

The points labeled by $1, \ldots, m$ represent the ends of the lines $v_1, \ldots, v_m$. Using this prescription, one can consecutively draw the diagrams for $P^{n_1}_{I_1}(v_1, \ldots, v_m)$, $P^{n_2}_{I_2}P^{n_1}_{I_1}(v_1, \ldots, v_m), \ldots, v$ (10). The diagram for $v$ has $m + 1$ external lines. The auxiliary points $1, \ldots, m$ are removed.

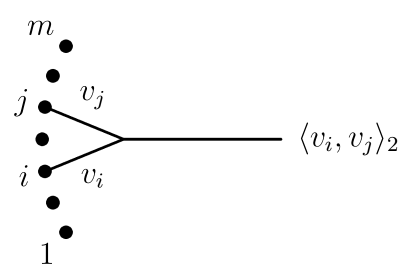


Figure 1. Diagram for $P^m_{ij}(v_1, \ldots, v_m)$.

For $v_1 = \ldots = v_m = w_0$ equation (9) reads

$$\langle w_0^m\rangle = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{s_1+\ldots+s_n=m} \frac{m!}{s_1! \ldots s_n!} \langle\langle w_0^{s_1}\rangle, \ldots, \langle w_0^{s_n}\rangle\rangle_n, \tag{12}$$

where $\langle w_0^r \rangle = \langle \underbrace{w_0, \ldots, w_0}_{r} \rangle$.

We find that a solution of equation (8) is given by

$$v = \langle e^{w_0} \rangle, \tag{13}$$

where

$$\langle e^{w_0} \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \langle w_0^k \rangle, \quad \langle w_0^0 \rangle = 0.$$

Indeed, substituting (13) in (8), we get

$$\sum_{m=2}^{\infty} \frac{1}{m!} \langle w_0^m \rangle = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{s_1+\ldots+s_n=m} \frac{1}{s_1! \ldots s_n!} \langle \langle w_0^{s_1} \rangle, \ldots, \langle w_0^{s_n} \rangle \rangle_n.$$

To conclude the proof, it remains to use (12).

Let

$$\xi = \xi_0 + \sum_{n=2}^{\infty} \frac{\epsilon_n}{n!} \xi^n$$

be an equation, where $\xi, \xi_0 \in \mathbb{R}$, $\epsilon_n$ is defined as 0 if $\langle v, \ldots, v \rangle_n = 0$, and otherwise $\epsilon_n = 1$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(\xi) = \xi - \sum_{n=2}^{\infty} \frac{\epsilon_n}{n!} \xi^n,$$

and let $g^{-1}(\xi) = \sum_{m=1}^{\infty} a_m \xi^m$ be the inverse function of $g$. Then

$$\langle e^{\xi} \rangle = \sum_{m=1}^{\infty} a_m \xi^m,$$

where $\langle \xi, \ldots, \xi \rangle_n = \xi^n$, and therefore the number of the descendants of $(v_1, \ldots, v_m)$ is given by $m! a_m$.

## 3 The Yang-Mills field equations

Let $x = (x^0, x^1, x^2, x^3)$ be space-time coordinates, $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ the metric tensor and $\Box = \partial^\mu \partial_\mu$. The pure Yang-Mills equations in the Lorentz gauge $\partial^\mu A_\mu = 0$ read [4]

$$\Box A_\nu + [A^\mu, (\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu])] - \partial^\mu [A_\mu, A_\nu] = 0, \tag{14}$$

where $A_\mu(x)$ is a non-Abelian gauge field. For $u_0 = 0, F_0 = 0, v = A_\nu dx^\nu$,

$$F_1(v) = \Box A_\nu dx^\nu, \quad F_2(v) = (\partial^\mu [A_\mu, A_\nu] - [A^\mu, (\partial_\nu A_\mu - \partial_\mu A_\nu)]) dx^\nu,$$
$$F_3(v) = -[A^\mu, [A_\mu, A_\nu]] dx^\nu$$

and $F_n(v) = 0, n \geq 4$, equations (14) are identified with (2).

The general solution $v_0 = A_{0\nu} dx^\nu$ of the free equation

$$\Box v = 0$$

is given by

$$v_0 = \frac{1}{4\pi} M(\psi) + \frac{1}{4\pi} \frac{\partial}{\partial t} (t M(\varphi)),$$

where

$$M(\mu) = \int_S \mu(x^1 + t\xi^1, x^2 + t\xi^2, x^3 + t\xi^3) d\sigma_\xi,$$

$\xi^1, \xi^2$ and $\xi^3$ are coordinates on the unit sphere $S$, $\sigma_\xi$ is the area element on $S$, $t = x^0$,

$$v_0(0, x^1, x^2, x^3) = \varphi(x^1, x^2, x^3), \quad \left. \frac{\partial v_0(t, x^1, x^2, x^3)}{\partial t} \right|_{t=0} = \psi(x^1, x^2, x^3).$$

A specific solution of the inhomogeneous equation

$$\Box v = -f$$

reads [1]

$$\mathcal{E}(f) = -\frac{1}{4\pi} \int_0^t \tau d\tau \int_S f(t - \tau, x^1 + \tau\xi^1, x^2 + \tau\xi^2, x^3 + \tau\xi^3) d\sigma_\xi.$$

We can rewrite equation (14) in the form (8)

$$A = A_0 + \sum_{n=2}^{\infty} \frac{1}{n!} \langle A, \ldots, A \rangle_n, \tag{15}$$

where $A = A_\nu dx^\nu$,$A_0 = A_{0\nu} dx^\nu$. Combining (15),(8) and (13), we get

$$A = \langle e^{A_0} \rangle.$$

For example, the $O(A_0^3)$ contribution in $A$ is given by

$$\frac{1}{6}\langle A_0, A_0, A_0 \rangle = \frac{1}{2}\langle\langle A_0, A_0\rangle_2, A_0\rangle_2 + \frac{1}{6}\langle A_0, A_0, A_0\rangle_3.$$

The diagram for $\langle\langle A_0, A_0\rangle_2, A_0\rangle_2$ is depicted in Figure 2. Here the Yang-Mills fields are represented by wavy lines.

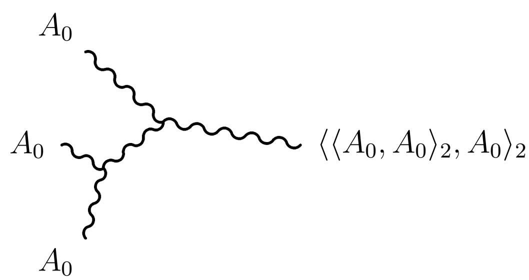


Figure 2. Diagram for $\langle\langle A_0, A_0\rangle_2, A_0\rangle_2$

# References


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